Free energy of disordered urn models in the canonical ensemble

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Urn models have been studied a lot because they are analytically tractable and contain rich physical phenomena, e.g., slow relaxation or condensation phenomena.[1–5] It has been revealed that the urn models are related to zero range processes and asymmetric simple exclusion processes[6] which are stochastic processes for studying nonequilibrium statistical physics. Furthermore, the urn models have been used in research fields of complex networks.[7,8]

Recently, the analytical treatment for disordered versions of the urn models have been developed.[5,8,9] Leuzzi and Ritort have analyzed a disordered urn model (disordered backgammon model) in the scheme of the grand canonical ensemble. On the other hand, in the scheme of the canonical ensemble, the replica method has been used in order to calculate the occupation distribution of the preferential urn model.[9] The replica method is a powerful tool for random systems, but still has an ambiguity for the mathematical validation. Furthermore, the analysis in ref. 9 has been based on replica symmetric solutions, so that it was not clear that the replica symmetric solution is adequate for the disordered urn model, although the solutions are in good agreement with numerical experiments.

In this short note, we calculate the free energy of the disordered urn model using the law of large numbers. It is revealed that the saddle point equation obtained by the usage of the law of large numbers is the same as that obtained by the replica method in ref. 9. Hence, we conclude that the replica symmetric solution is adequate for the disordered urn model. Furthermore, we point out the mathematical similarity of free energies between the urn models and the Random Field Ising Model (RFIM); this similarity gives an evidence that the replica symmetric solution of the urn models is exact.

Firstly, we give a brief explanation of a general urn model (Fig. 1). An Urn model consists of many urns and balls. We consider a system of M balls distributed among N urns; the number of balls contained in urn i is denoted by n_i, and hence $\sum_{i=1}^{N} n_i = M$. The density of the system is defined by $\rho = M/N$. There are two types of urn models: the Ehrenfest class and the Monkey class.[4] In the Ehrenfest class, balls within an urn are distinguishable. In contrast, the Monkey class has indistinguishable balls. These two types of urn models are treated in the similar analytical method, so that we discuss here only the case of the Ehrenfest class.

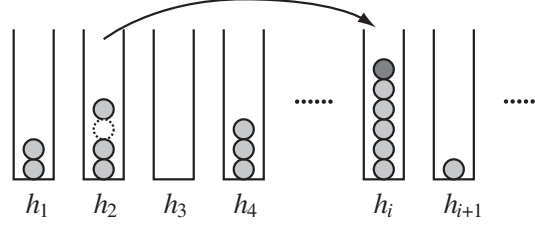[*]E-mail address: ohkubo@issp.u-tokyo.ac.jp

Fig. 1.   Illustration of the urn model. Each urn has a disorder parameter h_i.

The dynamics of the Ehrenfest urn model is briefly summarized as follows:

(1) Choose a ball at random.

(2) Select an urn with a transition probability u_i.

(3) Transfer the chosen ball to the selected urn i.

(4) Repeat the above procedures until the system reaches an equilibrium state.

Selecting an arbitrary transition probability u_i, one can construct various urn models suitable for own physical problems.

The total energy of the whole system is defined as

$$E(n_1, \ldots, n_N) = \sum_{i=1}^{N} E(h_i, n_i), \tag{1}$$

where $E(h_i, n_i)$ is an energy of urn i, and h_i the disorder parameter for urn i. It is assumed that the disorder parameter h_i is assigned to each urn by using a probability density $\phi(h)$. The partition function of the urn model is written as[4]

$$Z = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} p(h_1, n_1) \cdots p(h_N, n_N) \delta(\sum_{i=1}^{N} n_i, M), \tag{2}$$

where

$$p(h_i, n_i) = \frac{1}{n_i!} e^{-\beta E(h_i, n_i)} \tag{3}$$

is the Boltzmann weight attached to urn i, and β an inverse temperature. The Kronecker delta $\delta(\sum_{i=1}^{N} n_i, M)$ stems from the constraint of the total number of balls. Using the Boltzmann weight, the transition probability is defined as[6]

$$u_i = \frac{p(h_i, n_i - 1)}{p(h_i, n_i)}. \tag{4}$$

The aim of the present paper is to calculate the free energy in the thermodynamic limit by using the canonical partition function Z of eq. (2). In order to calculate the free energy, we rewrite the partition function Z as

$$Z = \oint \frac{dz}{2\pi} \frac{1}{z} \exp\left[-N\left\{\rho \ln z - \frac{1}{N} \sum_{i=1}^{N} \ln H(h_i, z)\right\}\right], \tag{5}$$

where we use the integral representation of the Kronecker delta

$$2i\pi\delta(m,n) = \oint dz\, z^{m-n-1} \tag{6}$$

and define

$$H(h,z) = \sum_{n=0}^{\infty} p(h,n)z^n. \tag{7}$$

Hence, the free energy of the disordered urn model in the configurational average is written by

$$\frac{1}{N}\langle F\rangle_{\{h\}} = \frac{1}{N}(-\frac{1}{\beta})\langle \ln Z\rangle_{\{h\}}$$

$$= -\frac{1}{\beta N}\left\langle \ln \oint \frac{dz}{2\pi}\frac{1}{z}\exp\left[-N\left\{\rho\ln z - \frac{1}{N}\sum_{i=1}^{N}\ln H(h_i,z)\right\}\right]\right\rangle_{\{h\}}, \tag{8}$$

where the configurational average is defined as

$$\langle A(h_1,\cdots,h_N)\rangle_{\{h\}}$$

$$= \int dh_1\phi(h_1)\cdots\int dh_N\phi(h_N)A(h_1,\cdots,h_N). \tag{9}$$

The law of large numbers is available in order to calculate the configurational average of eq. (8). Using the law of large numbers, $\frac{1}{N}\sum_{i=1}^{N}x_i = \langle x\rangle$ $(N\to\infty)$, the term $(\sum_{i=1}^{N}\ln H(h_i,z))/N$ in eq. (8) is replaced by $\langle \ln H(h,z)\rangle_h$. Hence, it becomes easy to take the configurational average. By using the saddle-point method, we get the free energy

$$\frac{1}{N}\langle F\rangle_{\{h\}} = -\frac{\rho}{\beta}\ln z_{\mathrm{s}} + \frac{1}{\beta}\langle \ln H(h,z_{\mathrm{s}})\rangle_h \tag{10}$$

and the saddle-point equation

$$\rho = \left\langle \frac{z_{\mathrm{s}}}{H(h,z_{\mathrm{s}})}\frac{d}{dz_{\mathrm{s}}}H(h,z_{\mathrm{s}})\right\rangle_h. \tag{11}$$

We note that the final result is consistent with the analytical results obtained by the replica method.[9] Hence, we conclude that the replica symmetric solution in ref. 9 is valid for the disordered urn model.

Finally, we will point out the mathematical similarity between the urn model and the RFIM. The RFIM consists of N Ising spins interacting through an infinite ranged exchange interaction. The Hamiltonian is given by

$$\mathcal{H} = -\frac{J}{N}\sum_{i<j}S_iS_j - \sum_i h_iS_i, \tag{12}$$

where J is the spin coupling constant, $\{S_i\}$ are spin variables, and $\{h\} = \{h_1,\cdots,h_N\}$ are

the external magnetic fields. The free energy of the RFIM is written as[10]

$$
\begin{aligned}
&\frac{1}{N}\langle F_{\mathrm{RFIM}}\rangle_{\{\boldsymbol{h}\}} \\
&= \frac{1}{\beta N}\left(\ln\sqrt{\frac{\beta J N}{2\pi}} - \frac{1}{2}\beta J\right) \\
&\quad - \frac{1}{\beta N}\left\langle\ln\int_{-\infty}^{\infty}dm\,\exp\left[-N\left\{\frac{1}{2}\beta J m^2 - \frac{1}{N}\sum_{i=1}^{N}\ln 2\cosh\beta(h_i + Jm)\right\}\right]\right\rangle_{\{\boldsymbol{h}\}},
\end{aligned}
$$

(13)

where β is the inverse temperature. For the RFIM, it has been revealed that the replica symmetric solution gives an exact result.[10] We can easily see the similarity of the free energy of the RFIM in eq. (13) and that of the urn model in eq. (8): the configurational average of a logarithm of an integral is taken, and the term depending on the disorder parameters is contained in the integral as the argument of the exponential function. We here remark a difference between the urn models and the RFIM. The variable of integration in eq. (13) is the magnetization, $m = \left(\sum_{i=1}^{N} S_i\right)/N$, and that in the disordered urn model is related to the macroscopic constraint $M = \sum_{i=1}^{N} n_i$ or $\rho = \left(\sum_{i=1}^{N} n_i\right)/N$. The macroscopic observable M (or ρ) in the disordered urn models is a constraint, therefore the observable is known in advance. In contrast, the magnetization m in the RFIM is unknown in advance and constructed by microscopic variables S_i with time. Although there are such differences between the urn models and the RFIM, the macroscopic observable constructed by microscopic variables (S_i or n_i) plays a key role in the both analysis.

In summary, we discussed a disordered version of a general urn model. The free energy and the saddle point equation were obtained by the usage of the law of large numbers, and it was clarified that the saddle point equation is the same as that of the replica symmetric solution. In addition, we pointed out the similarity between the urn models and the RFIM. Although we have not given the rigorous mathematical proof, we expect from the similarity that the replica symmetric solution describes exact results in the thermodynamic limit. We hope that the analogy with the RFIM will lead to better understanding of the disordered urn models.

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